Lecture 7: Wave eq. Pt 2

Boundary Problems.

- In our model of the string, we gave a length l and fixed the ends: $u(t,l) = u(t,0) = 0$. $t \geq 0$ } An example of Dirichlet Boundary Conditions

We may also describe the initial state of the string

$$u(0,x) = g(x) \qquad \frac{\partial u}{\partial t}(0,x) = h(x)$$

} Two derivatives for the equation means two pieces of data necessary

~ e.g. these can model a string being ~~plucked~~ in motion

- We assume $g(x) \in C^2([0,l])$ and $h \in C^1([0,l])$ to match $u \in C^2$

- Since h & g aren't on $\mathbb{R}$, we must extend them to $\mathbb{R}$ if we wish to apply our solution formula:

$\boxed{\text{Th}^{\underline{m}}\ 4.5}$ The wave equation $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ on $[0,l]$ with the above boundary and initial conditions admits a solution

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\,d\tau$$

Only if g & h extend to $\mathbb{R}$ as odd, $2l$-periodic ~~functions~~ functions $\tilde{g} \in C^2(\mathbb{R})$, $\tilde{h} \in C^1(\mathbb{R})$.

$\boxed{\text{Pf}}$ By the linearity of the wave equation (& homogeneity), we can consider the h & g terms separately. (How?)

Assume the term $\frac{1}{2}\left[g(x+ct) + g(x-ct)\right]$ is defined for all t & x and satisfies the boundary conditions on $[0,l]$ for all t. ie $x=0$

$$g(x+ct) + g(x-ct) = 0 = g(ct) + g(-ct) \qquad \text{for all } t \geq 0$$

Says g is odd. At $x = l$

$$g(l+ct) + g(l-ct) = 0 \qquad \text{for } t \geq 0$$

gives g is odd with respect to l: $g(l+ct) = -g(l-ct) = g(-l+ct)$

So that g is $2l$-periodic.

On $\frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\,d\tau = u(t,x)$, we see at $x=0$ $\frac{1}{2c}\int_{-ct}^{ct} h(\tau)\,d\tau = 0$

So, differentiating, $(h(ct) + h(-ct))c = 0$ so h is odd.

We note h is $2l$ periodic by doing the same at $x = l$. $\square$

ex.) Let $c=1$, $l=1$

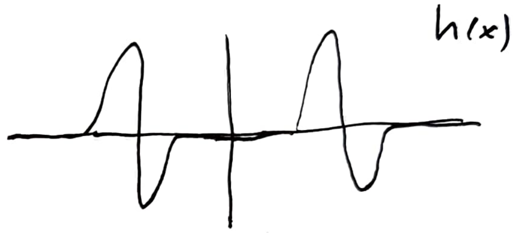
Suppose $u_\pm(x) = \frac{1}{2}g(x) \mp \frac{1}{2c}\int_0^x h(z)\,dz$ (so $u(x,t) = u_+(x-ct) + u_-(x+ct)$)

Let $u_+ = 0$ (no forward wave) and the left-propagating solution is u_-

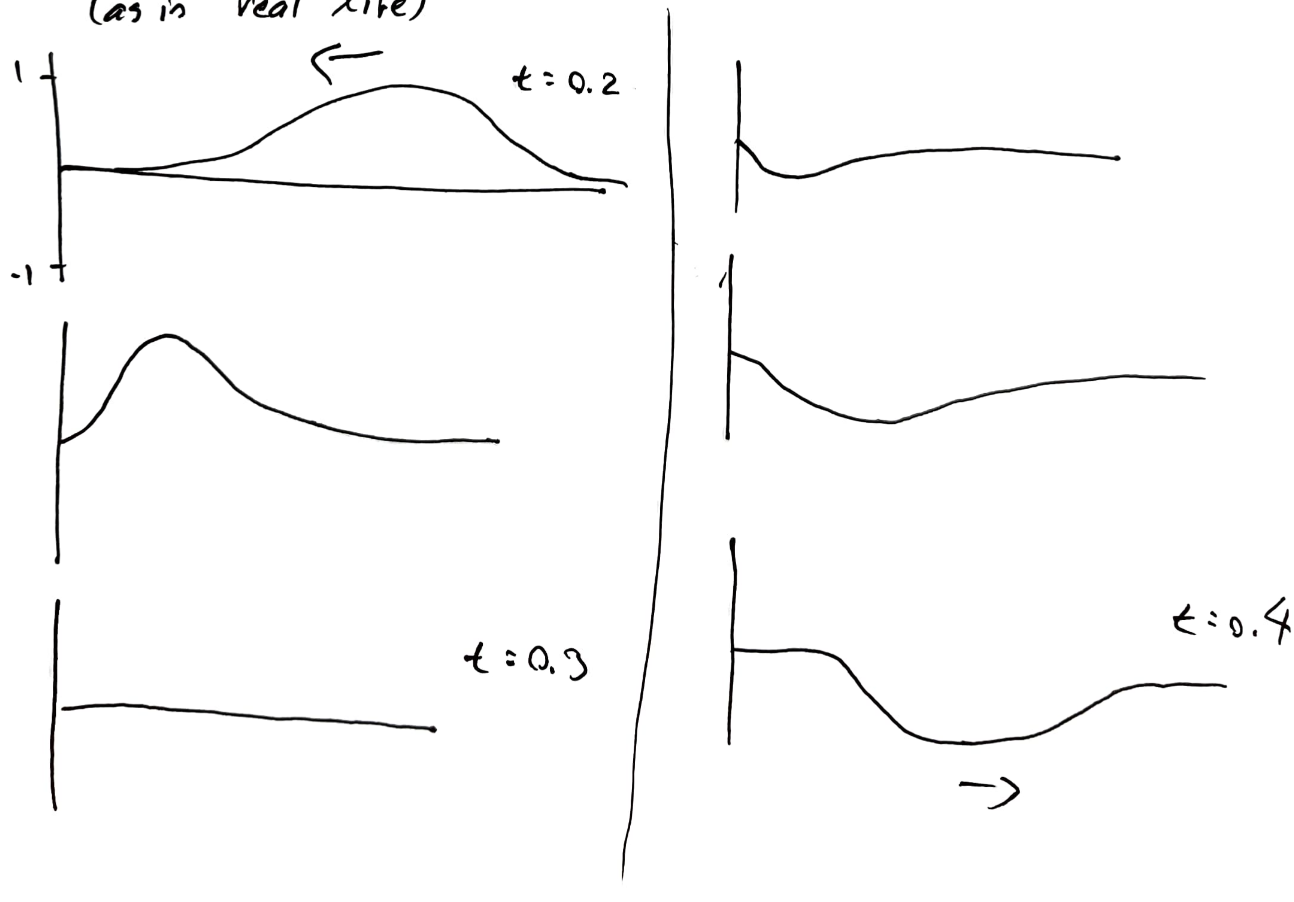
For small t, $u(t,x) = u_-(x+t)$ as we solved for. What happens when the bump hits the boundary $x=0$?

- Since $u(t,x) = u_-(x+t)$, $g(x) = u(0,x) = u_-(x)$ and $h(x) = \frac{\partial u_-}{\partial x}(x)$.

by extending as in thm. 4.5



Computing the solution, our bump "bounces" off the boundary (as in real life)



Forcing Terms

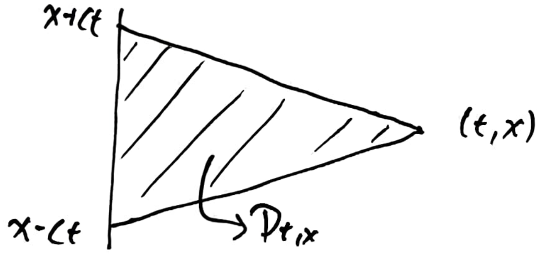
- We previously used the homogeneous wave equation $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ in accordance with having no external forces. Let us now introduce an external force such as a bow on a violin.

(A)
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(t,x) \qquad u(0,x) = g(x), \quad \frac{\partial u}{\partial t}(0,x) = h(x)$$

- To solve this equation, we will manipulate our homogeneous solution in a version of variation of parameters called Duhamel's Principle.
  - ⤳ to focus on this, we consider the domain $\mathbb{R}$ and set initial conditions to zero: $g(x) = h(x) = 0$

- Define the "Domain of Dependence"

$$D_{t,x} = \{ (s,x') \in \mathbb{R}_+ \times \mathbb{R} : \ x - c(t-s) \le x' \le x + c(t-s) \}$$



⤳ The "inverted wave cone" influencing the solution.

⤳ "information travels at speed c"

$\boxed{\text{Th}^{\underline{m}}\ 4.7}$ For $f \in C^1(\mathbb{R})$, the unique solution of (A) satisfying $g(x) = h(x) = 0$ is given by

$$u(t,x) = \frac{1}{2c} \int_{D_{t,x}} f(s,x')\, dx'\, ds$$

$\boxed{\text{Pf}}$ For each $s \ge 0$, let $\eta_s(t,x)$ be the solution of the homogeneous wave eqn. for $t \ge s$ with initial conditions

$$\eta_s(t,x)\big|_{t=s} = 0 \qquad \frac{\partial \eta_s}{\partial t}(t,x)\Big|_{t=s} = f(s,x)$$

By shifting $t \to t-s$, our previous formula gives

$$\eta_s(t,x) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(s,x')\,dx'$$

We claim that the solution of (A) is given by

$$u(t,x) = \int_0^t \eta_s(t,x)\,ds = \int_0^t \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(s,x')\,dx'\,ds$$

•) First, $u(0,x) = \int_0^0 \eta_s \, ds = 0$

$\cancel{\frac{\partial u}{\partial t}(0,x) = \eta_t(t,x)|_{t=0} = 0}$

$$\frac{\partial u}{\partial t}(t,x) = \eta_s(t,x)\Big|_{s=t} + \underbrace{\int_0^t \frac{\partial \eta_s}{\partial t}(t,x)\,ds}_{\text{Leibniz Rule}} \quad \text{so}$$

$$\frac{\partial u}{\partial t}(0,x) = \eta_0(0,x) + \int_0^0 \cdots ds = 0$$

•) Next, we check the PDE:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \eta_s}{\partial t}(t,x)\Big|_{t=s} + \int_0^t \frac{\partial^2 \eta_s}{\partial t^2}(t,x)\,ds$$

$$\left(+ \frac{\partial \eta_s}{\partial t}(t,x)\Big|_{t=s} \text{ which is } 0\right)$$

By the initial conditions for η_s, $\frac{\partial \eta_s}{\partial t}(t,x)\Big|_{t=s} = f(t,x)$

as for the second term

$$\int_0^t \frac{\partial^2 \eta_s}{\partial t^2}(t,x)\,ds = \int_0^t c^2 \frac{\partial^2 \eta_s}{\partial x^2}(t,x)\,ds = c^2 \frac{\partial^2 u}{\partial x^2}$$

s.t. $\frac{\partial^2 u}{\partial t^2} = f + c^2 \frac{\partial^2 u}{\partial x^2}$ as desired.

•) Lastly, we show uniqueness: if u_1 and u_2 solve (A),

$w = u_1 - u_2$ solves $\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} = 0$, $w(0,x) = 0$, $\frac{\partial w}{\partial t}(0,x) = 0$.

By our thm. for the homogeneous equation, $w = 0$. $\square$

• Rmk: to solve

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f \\ u(0,x) = g \\ \frac{\partial u}{\partial t}(0,x) = h \end{cases}$$

set $u = v + w$ for v solving (A) and w solving

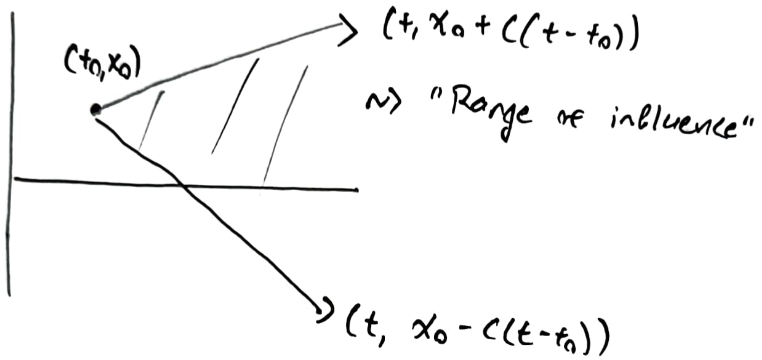
$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} = 0 \\ w(0,x) = g \\ \frac{\partial w}{\partial t}(0,x) = h \end{cases}$$

⇝ The Domain of Dependence still applies b/c

$$w(t,x) = \cancel{\tfrac{1}{2}[g(t,x)} \; \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds$$

relies on g, h in the triangle $D_{t,x}$.

- For a given (t_0, x_0), the Domain of Dependence means that there is a "range of influence" of (t_0, x_0) of ~~[illegible]~~ points at which (t_0, x_0) influences the solution $u(t,x)$



ex.) Consider a string of length ℓ with propagation speed $c=1$. Let $f(t,x) = \cos(\omega t)\sin(\omega_0 x)$ for $\omega_0 = \pi/\ell$, $\omega > 0$.

Since $\sin(\omega_0 x)$ is odd & 2ℓ-periodic, $f(t,x)$ satisfies the conditions of thm 4.5 for each t. Set $g = h = 0$.

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \cos(\omega s)\sin(\omega_0 x')\,dx'\,ds$$

$$= \frac{1}{2\omega_0}\int_0^t \underbrace{\cos(\omega s)\left[\cos(\omega_0(x-t+s)) - \cos(\omega_0(x+t-s))\right]}_{I}\,ds$$

$$I = \cos(\omega s)\Big[\cos(\omega_0 x)\cos(\omega_0(s-t)) - \sin(\omega_0 x)\sin(\omega_0(s-t)) - \cos(\omega_0 x)\cos(\omega_0(t-s)) + \sin(\omega_0 x)\sin(\omega_0(t-s))\Big]$$

$$= 2\cos(\omega s)\sin(\omega_0 x)\sin(\omega_0(t-s))$$

or

$$u(t,x) = \frac{\sin(\omega_0 x)}{\omega_0}\int_0^t \cos(\omega s)\sin(\omega_0(t-s))\,ds$$

computations...

expand $\cos(\omega s)\sin(\omega_0(t-s)) = \frac{1}{2}\left[\sin(\omega_0(t-s) + \omega s) - \sin(\omega s - \omega_0(t-s))\right]$

Split into 2 integrals, u-sub

$$\frac{1}{2}\int_0^t \sin(\omega_0 t + s(\omega - \omega_0))\,ds - \frac{1}{2}\int_0^t \sin(s(\omega+\omega_0) - \omega_0 t)\,ds$$

$$\cancel{= \frac{1}{2}\Big[-\frac{1}{\omega-\omega_0}\cos(\omega_0 t + s(\omega-\omega_0))}$$

$$= \frac{1}{2}\left[-\frac{1}{\omega-\omega_0}\cos(\omega_0 t + s(\omega-\omega_0)) + \frac{1}{\omega+\omega_0}\cos(s(\omega+\omega_0) - \omega_0 t)\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{\omega-\omega_0}\left(\cos(\omega_0 t)-\cos(\omega t)\right) - \frac{1}{\omega+b}\left(\cos(\omega_0 t)-\cos(\omega t)\right)\right]$$

$\vdots$ Simplify

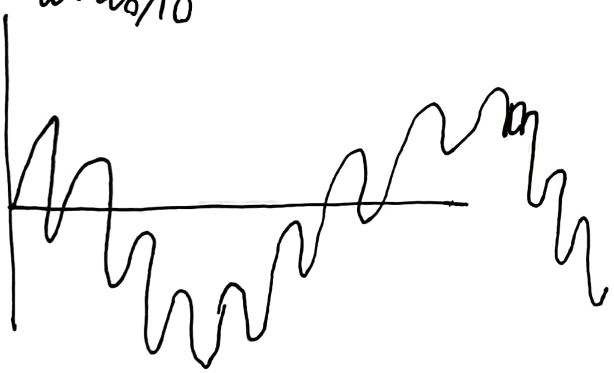
For $\omega \neq \omega_0$, we obtain

$$u(t,x) = \frac{\sin(\omega_0 x)}{\omega_0^2-\omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right] \qquad \text{assume } \omega << \omega_0$$

- dual oscillation ~ large scale on period $1/\omega$ given by the "low driving frequency" & small scale on period $1/\omega_0$ depending on l

For $\omega = \omega_0$, $u(t,x) = \frac{t}{2\omega_0} \sin(\omega_0 x)\sin(\omega_0 t)$ ~ growing amplitude akin to <u>resonance</u> (absorbing energy from the driving source continually)

$\omega = \omega_0/10$



$\omega = \omega_0$